COMODULES OF $U_q(sl_2)$ AND MODULES OF $SL_q(2)$ VIA QUIVER

XIAO-WU CHEN[A] AND PU ZHANG[B,*]

[A]Department of Mathematics
University of Science and Technology of China
Hefei 230026, Anhui, P. R. China

[B]Department of Mathematics
Shanghai Jiao Tong University
Shanghai 200240, P. R. China

xwchen@mail.ustc.edu.cn
pzhang@sjtu.edu.cn

Abstract. The aim of this paper is to construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver, where q is not a root of unity.

By embedding $U_q(sl_2)$ into the path coalgebra $k\mathcal{D}^c$, where $\mathcal{D}$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths in the quiver $\mathcal{D}$; this special basis enable us to describe the category of $U_q(sl_2)$-comodules by certain representations of $\mathcal{D}$; and this description further permits us to construct a class of modules of $SL_q(2)$, from certain representations of $\mathcal{D}$, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

1. Introduction

Drinfeld [Dr] has established a duality, between the quantized enveloping algebra $U_q(sl_2)$ and the quantum deformation $SL_q(2)$ of the regular function ring on SL_2 (see [K], VII). This has been extended between $U_q(sl_n)$ and $SL_q(n)$ by Takeuchi [T]. Therefore, any $U_q(sl_n)$-comodule (resp. $SL_q(n)$-comodule) can be endowed with a $SL_q(n)$-module structure (resp. a $U_q(sl_n)$-module), in a canonical way (see e.g. (5.1) below). However, this duality does not give $U_q(sl_n)$-comodules (resp. $SL_q(n)$-comodules) from $SL_q(n)$-modules (resp. $U_q(sl_n)$-modules).

Modules of $U_q(\mathrm{g})$ have been extensively studied (see e.g. [L], [Ro], [J]), and it depends on q: when q is not a root of unity, any finite-dimensional module is semi-simple, and the finite-dimensional simple module is a deformation of a finite-dimensional simple g-module. Another thing of $U_q(\mathrm{g})$ which depends on q is its coradical filtration ([Bo], [CMus], [M1], [Mü]): when q is not a root of unity, the graded coalgebra $U_q(\mathrm{g})$ is coradically graded.

* The corresponding author.

Supported in part by the National Natural Science Foundation of China (Grant No. 10271113 and No. 10301033) and the Doctoral Foundation of the Chinese Education Ministry. The first named author is supported by the AsiaLink project "Algebras and Representations in China and Europe" ASI/B7-301/98/679-11.

The study of $SL_q(n)$-comodules can be also founded, e.g. in [PW], [CK] (see also [G]). However there are few works on $U_q(sl_n)$-comodules. A possible reason of this lack might be that there are no proper tools to construct $U_q(sl_n)$-comodules. The aim of the present paper is to understand the $U_q(sl_2)$-comodules by using the quiver techniques.

In the representation theory of algebras, quiver is a basic technique (see [ARS], [Rin]). Recently, it also shows powers in studying coalgebras and Hopf algebras. For example, one can construct path coalgebras of quivers, define the Gabriel quiver of a coalgebra, and embed a pointed coalgebra into the path coalgebra of its Gabriel quiver (see [CMont], [M3], [CHZ]); after this embedding one can expect to study the comodules of the coalgebra by certain locally-nilpotent representations of the quiver (see [C]); and this makes it possible to see the morphisms, the extensions, and even the Auslander-Reiten sequences (see e.g. [Sim]). One can also start from the Hopf quivers of groups to construct non-commutative, non-cocommutative pointed Hopf algebras (see [CR]); this makes it possible to classify some Hopf algebras by quivers, whose bases can be explicitly given (see e.g. [CHYZ], [OZ]).

Inspired by these ideas, in this paper, we construct comodules of $U_q(sl_2)$ and modules of $SL_q(2)$ via quiver, where q is not a root of unity. By embedding the quantized algebra $U_q(sl_2)$ into the path coalgebra $k\mathcal{D}^c$, where $\mathcal{D}$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra, we obtain a basis of $U_q(sl_2)$ in terms of combinations of paths in the quiver $\mathcal{D}$ (Theorem 3.5); this special basis enables us to describe the category of $U_q(sl_2)$-comodules by certain locally-nilpotent representations of $\mathcal{D}$ (Theorem 4.3); in particular, we can list all the indecomposable Schurian comodules of $U_q(sl_2)$ (Theorem 4.7); and this description further permits us to construct a class of modules of the quantum special linear group $SL_q(2)$, from certain locally-nilpotent representations of $\mathcal{D}$, via the duality between $U_q(sl_2)$ and $SL_q(2)$ (Theorem 5.2).

2. **Preliminaries**

Throughout this paper, let k denote a field of characteristic zero, and q a non-zero element in k with $q^2 \neq 1$. For a k-space V, let V^* denote the dual space. Denote by $\mathbb{Z}$ and $\mathbb{N}_0$ the sets of integers and of non-negative integers, respectively.

2.1. By definition $U_q(sl_2)$ is an associative k-algebra generated by E, F, K, K^{-1}, with relations (see e.g. [K], p.122, or [J], p. 9)

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1,\\
&KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F,\\
&[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.
\end{aligned}$$

Then $U_q(sl_2)$ has a Hopf structure with (see e.g. [K], p.140)

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = \varepsilon(K^{-1}) = 1,$$
$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -EK^{-1}, \; S(F) = -KF.$$

Note that $U_q(sl_2)$ is a Noetherian algebra without zero divisors, and it has a basis $\{K^l E^i F^j \mid i, j \geq 0, l \in \mathbb{Z}\}$ (see e.g. [K], p.123).

By definition $SL_q(2)$ is an associative k-algebra generated by a, b, c, d, with relations (see e.g. [K], p.84)

$$ba = qab, \quad db = qbd, \quad ca = qac, \quad dc = qcd, \quad bc = cb,$$
$$ad - da = (q^{-1} - q)bc, \quad da - qbc = 1.$$

Then $SL_q(2)$ has a Hopf structure with (see e.g. [K], p.84)

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$
$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$
$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0,$$
$$S(a) = d, \quad S(b) = -qb, \quad S(c) = -q^{-1}c, \quad S(d) = a.$$

2.2. By definition a duality between two Hopf algebras U and H is an algebra map $\psi : H \longrightarrow U^*$, such that $\phi : U \longrightarrow H^*$ is also an algebra map and has the property

$$\psi(x)(S_U(u)) = \phi(u)(S_H(x))$$

for all $u \in U, x \in H$, where ϕ is defined by

$$\phi(u)(x) = \psi(x)(u),$$

and S_U and S_H are respectively the antipodes of U and H.

Suppose that there exists a duality between U and H. Then there also exists a duality between H and U; and each U-comodule can be endowed with an H-module structure, and also each H-comodule can be endowed with a U-module.

We have the following well-known duality between $U_q(sl_2)$ and $SL_q(2)$. See Theorem VII.4.4 in [K].

Lemma 2.3. *There is a unique algebra map $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$ such that*

$$\psi(a)(K^l E^i F^j) = \delta_{i,0}\delta_{j,0}q^l + \delta_{i,1}\delta_{j,1}q^l, \qquad \psi(b)(K^l E^i F^j) = \delta_{i,1}\delta_{j,0}q^l,$$
$$\psi(c)(K^l E^i F^j) = \delta_{i,0}\delta_{j,1}q^{-l}, \qquad \psi(d)(K^l E^i F^j) = \delta_{i,0}\delta_{j,0}q^{-l},$$

where $\delta_{i,j}$ is the Kronecker symbol. This ψ is a duality between $U_q(sl_2)$ and $SL_q(2)$.

Note that such a ψ is not injective. This duality was essentially introduced in [Dr], and has been extended to be a duality between $U_q(sl_n)$ and $SL_q(n)$ in [T].

2.4. A quiver $Q = (Q_0, Q_1, s, t)$ is a datum, where Q is an oriented graph with Q_0 the set of vertices and Q_1 the set of arrows, s and t are two maps from Q_1 to Q_0, such that $s(a)$ and $t(a)$ are respectively the starting vertex and terminating vertex of $a \in Q_1$. A path p of length l in Q is a sequence $p = a_l \cdots a_2 a_1$ of arrows $a_i,\ 1 \leq i \leq l$, such that $t(a_i) = s(a_{i+1})$ for $1 \leq i \leq l-1$. A vertex is regarded as a path of length 0. Denote by $s(p)$ and $t(p)$ the starting vertex and terminating vertex of p, respectively. Then $s(p) = s(a_1)$ and $t(p) = t(a_l)$. If both Q_0 and Q_1 are finite sets, then Q is called a finite quiver. We will not restrict ourselves to finite quivers, but we assume the quivers considered are countable (i.e., both Q_0 and Q_1 are countable sets). For quiver method to representations of algebras we refer to [ARS] and [Rin].

Given a quiver Q, define the path coalgebra kQ^c (see [CMon]) as follows: the underlying space has a basis the set of all paths in Q, and the coalgebra structure is given by

$$\Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha$$

and

$$\varepsilon(p) = 0 \;\text{ if }\; l \geq 1, \quad \text{and} \quad \varepsilon(p) = 1 \;\text{ if }\; l = 0$$

for each path p of length l.

2.5. By a graded coalgebra we mean a coalgebra C with decomposition $C = \bigoplus\limits_{n \geq 0} C(n)$ of k-spaces such that

$$\Delta(C(n)) \subseteq \sum_{i+j=n} C(i) \otimes C(j), \qquad \varepsilon(C(n)) = 0, \qquad \forall\ n \geq 1.$$

Let C be a coalgebra. Following [Sw], the wedge of two subspaces V and W of C is defined to be the subspace

$$V \wedge W := \{c \in C \mid \Delta(c) \in V \otimes C + C \otimes W\}.$$

Let C_0 be the coradical of C, i.e., C_0 is the sum of all simple subcoalgebras of C. Define $C_n := C_0 \wedge C_{n-1}$ for $n \geq 1$. Then $\{C_n\}_{n \geq 0}$ is called the coradical filtration of C.

Recall that a graded coalgebra $C = \bigoplus\limits_{n \geq 0} C(n)$ is said to be coradically graded, provided that $\{C_n := \bigoplus\limits_{i \leq n} C(i)\}_{n \geq 0}$ is exactly the coradical filtration of C. It was proved in [CMus], 2.2, that a graded coalgebra $C = \bigoplus\limits_{n \geq 0} C(n)$ is coradically graded if and only if $C_0 = C(0)$ and $C_1 = C(0) \oplus C(1)$.

2.6. Let M be a C-C-bicomodule over a coalgebra C. Denote by $\mathrm{Cot}_C(M)$ the corresponding cotensor coalgebra (see [D] for the definition and basic properties). This is a graded coalgebra with 0-th component C and 1-th component M. By Proposition 11.1.1 in [Sw], the coradical of $\mathrm{Cot}_C(M)$ is contained in C. It follows that $\mathrm{Cot}_C(M)$ is coradically graded if and only if C is cosemisimple.

Note that a path coalgebra kQ^c is graded with the length grading, and it is coradically graded, and $kQ^c \simeq \mathrm{Cot}_{kQ_0}(kQ_1)$ (see [CMon], or [CR]).

We need the following observation.

Proposition 2.7. *Let* $C = \bigoplus_{n\geq 0} C(n)$ *be a graded coalgebra. Then*

(i) There is a unique graded coalgebra map $\theta : C \longrightarrow \mathrm{Cot}_{C(0)}(C(1))$ *such that* $\theta|_{C(i)} = Id$ *for* $i = 0, 1$.

(ii) $\theta(x) = \pi^{\otimes n+1} \circ \Delta^n(x)$ *for all* $x \in C(n+1)$ *and* $n \geq 1$, *where* $\pi : C \longrightarrow C(1)$ *is the projection, and* $\Delta^n = (Id \otimes \Delta^{n-1}) \circ \Delta$ *for all* $n \geq 1$, *with* $\Delta^0 = Id$.

(iii) If C *is coradically graded, then* θ *is injective.*

(iv) If $C(0)$ *is cosemisimple, and* θ *is injective, then* C *is coradically graded.*

Proof Clearly, $C(0)$ is a subcoalgebra and $C(1)$ is naturally a $C(0)$-$C(0)$-bicomodule, and hence we have the corresponding cotensor coalgebra $\mathrm{Cot}_{C(0)}(C(1))$. The statements (i) and (ii) follow from the universal property of a cotensor coalgebra (see e.g. [Rad], or [CR]).

For the statement (iii), if C is coradically graded, then $C_1 = C(0) \oplus C(1)$. It follows that $\theta|_{C_1}$ is injective, and hence θ is injective, by a theorem due to Heynemann and Radford (see e.g. [M2], 5.3.1).

If $C(0)$ is cosemisimple, then $\mathrm{Cot}_{C(0)}(C(1))$ is coradically graded. The injectivity of θ implies that C is a graded subcoalgebra of $\mathrm{Cot}_{C(0)}(C(1))$. Thus C is also coradically graded. ■

2.8. Consider a special case of Proposition 2.7 where $C(0)$ is a group-like coalgebra (i.e., it has a basis consisting of group-like elements; or equivalently, $C(0)$ is cosemisimple and pointed). In this case we have $C(0) = kG(C)$, where

$$G(C) := \{g \in C \mid \Delta(g) = g \otimes g, \varepsilon(g) = 1\}.$$

Since $C(1)$ is a $C(0)$-$C(0)$-bicomodule, it follows that

$$C(1) = \bigoplus_{g,h \in G} {}^hC(1)^g,$$

where ${}^hC(1)^g = \{c \in C(1) \mid \Delta(c) = c \otimes g + h \otimes c\}$. Define a quiver $Q = Q(C)$ as follows: the set of vertices is G, and there are exactly t_{gh} arrows from vertex g to vertex h, where $t_{gh} = \dim_k {}^hC(1)^g$. Then by the universal property of a cotensor coalgebra (and hence of a path coalgebra), there is a coalgebra isomorphism $\mathrm{Cot}_{C(0)}(C(1)) \simeq kQ^c$, by identifying the elements of $G(C)$ with the vertices of Q and a basis of ${}^hC(1)^g$ with the arrows from g to h.

Note that the quiver $Q(C)$ is in general not the Gabriel quiver of C. If the graded coalgebra $C = \oplus_{n\geq 0} C(n)$ is coradically graded, then $Q(C)$ is exactly the Gabriel quiver of C. For the equivalent definitions of the Gabriel quiver of a coalgebra we refer to [CHZ], Section 2 (see also [CMon], [M3], and [Sim]). By Proposition 2.7 we have

Corollary 2.9. *Assume that* $C = \bigoplus_{n\geq 0} C(n)$ *is a graded coalgebra with* $C(0)$ *group-like. Let* $Q(C)$ *be the quiver associated to* C *defined as above. Then*

(i) There is a graded coalgebra map $\theta: \ C \longrightarrow kQ(C)^c$.

(ii) θ *is injective if and only if* C *is coradically graded. In this case,* $Q(C)$ *is exactly the Gabriel quiver of* C.

3. $U_q(sl_2)$ as a subcoalgebra of a path coalgebra

In this section, we embed $U_q(sl_2)$ into the path coalgebra of the Gabriel quiver $\mathcal{D}$ of $U_q(sl_2)$, and then give a set of basis of $U_q(sl_2)$ in terms of combinations of paths in $\mathcal{D}$, where q is not a root of unity.

Although bases of $U_q(sl_2)$ are already available, but this new set of basis of $U_q(sl_2)$ given here, which is in terms of combinations of paths in $\mathcal{D}$, will enable us to describe the category of $U_q(sl_2)$-comodules, in terms of k-representations of the quiver $\mathcal{D}$.

3.1. For each non-negative integer n, let $C(n)$ be the subspace of $U_q(sl_2)$ with basis the set $\{K^lE^iF^j \mid i, j \in \mathbb{N}_0, i+j = n, \ l \in \mathbb{Z}\}$. Then

$$U_q(sl_2) = \bigoplus_{n\geq 0} C(n)$$

is a graded coalgebra (see for example Proposition VII.1.3 in [K]) with

$$G(U_q(sl_2)) = \{K^l \mid l \in \mathbb{Z}\}, \quad \text{and} \quad C(0) = \bigoplus_{l\in\mathbb{Z}} kK^l.$$

We have in $C(1)$

$$\Delta(K^{l-1}E) = K^{l-1} \otimes K^{l-1}E + K^{l-1}E \otimes K^l,$$
$$\Delta(K^lF) = K^{l-1} \otimes K^lF + K^lF \otimes K^l.$$

Note that $C(1)$ has a set of basis $\{ \ K^lE, \ K^lF \mid \ l \in \mathbb{Z}\}$;

$${}^{K^{l_2}}C(1)^{K^{l_1}} = 0 \ \text{ for } \ (l_1, l_2) \neq (l, l-1), \quad l \in \mathbb{Z},$$

and that for each $l \in \mathbb{Z}$ we have

$${}^{K^{l-1}}C(1)^{K^l} = kK^{l-1}E \oplus kK^lF, \quad l \in \mathbb{Z}.$$

Therefore, the quiver of $U_q(sl_2)$ as defined in 2.8 is of the form

$$\cdots\cdots\bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet \rightrightarrows \bullet\cdots\cdots$$

We will denote this quiver by $\mathcal{D}$ in this paper.

3.2. We fix some notations. Index the vertices of $\mathcal{D}$ by integers, i.e., $\mathcal{D}_0 = \{e_l \mid l \in \mathbb{Z}\}$; there are two arrows from e_l to e_{l-1} for each integer l. Put $I = \{1, -1\}$ and let I^n be the Cartesian product (understand $I^0 := \{0\}$). Define $\mathcal{I} = \bigcup\limits_{n\geq 0} I^n$. For each

$v \in \mathcal{I}$, define $|v| = n$ if $v \in I^n$. Write v as $v = (v_1, \cdots, v_n)$, where $v_j = 1$ or -1 for each j. For any integer l and $v \in \mathcal{I}$, define

$$P_l^{(v)} = a_{|v|} \cdots a_2 a_1$$

to be the concatenated path in $\mathcal{D}$ starting at e_l of length $|v|$, where the arrow a_j is the upper arrow if $v_j = 1$, and the lower one if otherwise, $1 \le j \le |v|$.

For example, $P_l^{(0)}$ is understood to be the vertex e_l; $P_l^{(1)}$ (resp. $P_l^{(-1)}$) is the upper (resp. lower) arrows starting at the vertex e_l in $\mathcal{D}$. Clearly,

$$\{P_l^{(v)} = P_{l-|v|+1}^{(v_{|v|})} \cdots P_{l-1}^{(v_2)} P_l^{(v_1)} \mid l \in \mathbb{Z}, v \in \mathcal{I}\}$$

is the set of all paths in $\mathcal{D}$.

As an application of Corollary 2.9 we have

Lemma 3.3. *There is a unique graded coalgebra map $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$ such that $\theta(K^l) = e_l$, $\theta(K^{l-1}E) = P_l^{(1)}$, and $\theta(K^l F) = P_l^{(-1)}$, for each integer l.*

Moreover, if q is not a root of unity, then θ is injective. In this case, $\mathcal{D}$ is the Gabriel quiver of the coalgebra $U_q(sl_2)$.

Proof The existence of θ follows directly from Corollary 2.9, and the uniqueness follows from the universal property of a path coalgebra. Note that if q is not a root of unity, then the graded coalgebra $U_q(sl_2) = \bigoplus_{n \ge 0} C(n)$ is coradically graded (see [M1], or [M2], Question 5.5.6). ■

3.4. For $v \in I^n \subset \mathcal{I}$, put

$$T_v := \{t \mid 1 \le t \le n, \quad v_t = 1\}, \quad \chi(v) := q^{2\sum_{t \in T_v} t}, \quad \text{if } n \ge 1,\ T_v \ne \emptyset;$$
$$\chi(v) := 1, \qquad \text{otherwise.}$$

For each $l \in \mathbb{Z},\ n \in \mathbb{N}_0,\ 0 \le i \le n$, set

$$b(l, n, i) := \sum_{v \in I^n,\ |T_v| = i} \chi(v) P_l^{(v)} \in k\mathcal{D}^c.$$

For example, we have

$$\begin{aligned} &b(l,0,0) = e_l, \qquad b(l,1,0) = P_l^{(-1)}, \qquad b(l,1,1) = q^2 P_l^{(1)}, \\ &b(l,2,0) = P_l^{(-1,-1)}, \quad b(l,2,2) = q^6 P_l^{(1,1)}, \\ &b(l,2,1) = q^2 P_l^{(1,-1)} + q^4 P_l^{(-1,1)}. \end{aligned}$$

The main theorem of this section is

Theorem 3.5. *Assume that q is a not a root of unity. Then as a coalgebra $U_q(sl_2)$ is isomorphic to the subcoalgebra of $k\mathcal{D}^c$ with the set of basis*

$$\{b(l, n, i) \mid 0 \le i \le n,\ n \in \mathbb{N}_0, \quad l \in \mathbb{Z}\}.$$

For a non-zero element q in k, and non-negative integers $n \geq m$, the Gaussian binomial coefficient is defined to be

$$\binom{n}{m}_q = \frac{n!_q}{m!_q(n-m)!_q}$$

where $n!_q := 1_q 2_q \cdots n_q, \;\; 0!_q := 1, \;\; n_q := 1 + q + \cdots + q^{n-1}$.

Given a positive integer n, and two vectors $s = (s_0, s_1, \ldots, s_{n-1}), r = (r_0, r_1, \cdots, r_{n-1}) \in \mathbb{N}_0^n$ with the property

$$s_0 \geq s_1 \geq \cdots \geq s_{n-1}, \quad r_0 \geq r_1 \geq \cdots \geq r_{n-1},$$

set

$$c(s,r) := \binom{s_0}{s_1}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2} \binom{r_0}{r_1}_{q^{-2}} \cdots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}} q^{2\sum\limits_{t=1}^{n-1} r_t(s_{t-1}-s_t)}.$$

Lemma 3.6. *Put $E' := K^{-1}E \in U_q(sl_2)$. Then for any non-negative integers i and j, with $n := i + j \geq 1$, we have*

$$\Delta^{n-1}(K^l E'^i F^j) = \sum_{s,r} c(s,r) \quad K^{l-s_1-r_1} E'^{s_0-s_1} F^{r_0-r_1} \otimes \cdots$$
$$\otimes K^{l-s_{n-1}-r_{n-1}} E'^{s_{n-2}-s_{n-1}} F^{r_{n-2}-r_{n-1}} \otimes K^l E'^{s_{n-1}} F^{r_{n-1}}$$

where the sum runs over all the r and s with $s_0 = i$ and $r_0 = j$.

Proof It suffices to prove the formula for $n \geq 2$. Note that

$$\Delta(E'^i) = \Delta(E')^i = (K^{-1} \otimes E' + E' \otimes 1)^i$$
$$= \sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1}.$$

So

$$\Delta^{n-1}(E'^i) = (Id \otimes \Delta^{n-2})(\sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes E'^{s_1})$$
$$= \sum_{s_1=0}^{i} \binom{i}{s_1}_{q^2} K^{-s_1} E'^{i-s_1} \otimes \Delta^{n-2}(E'^{s_1}).$$

By induction we have

$$\Delta^{n-1}(E'^i) = \sum_{0 \leq s_{n-1} \leq s_{n-2} \leq \cdots \leq s_1 \leq i} \binom{i}{s_1}_{q^2} \binom{s_1}{s_2}_{q^2} \cdots \binom{s_{n-2}}{s_{n-1}}_{q^2}$$
$$K^{-s_1} E'^{i-s_1} \otimes K^{-s_2} E'^{s_1-s_2} \otimes \cdots \otimes K^{-s_{n-1}} E'^{s_{n-2}-s_{n-1}} \otimes E'^{s_{n-1}}.$$

Similarly, we have

$$\Delta^{n-1}(F^j) = \sum_{0\le r_{n-1}\le r_{n-2}\le\cdots\le r_1\le j} \binom{j}{r_1}_{q^{-2}} \binom{r_1}{r_2}_{q^{-2}} \cdots \binom{r_{n-2}}{r_{n-1}}_{q^{-2}}$$
$$K^{-r_1}F^{j-r_1} \otimes K^{-r_2}F^{r_1-r_2} \otimes \cdots \otimes K^{-r_{n-1}}F^{r_{n-2}-r_{n-1}} \otimes F^{r_{n-1}}.$$

Now the formula follows from $\Delta^{n-1}(K^lE'^iF^j) = \Delta^{n-1}(K^l)\Delta^{n-1}(E'^i)\Delta^{n-1}(F^j)$ and the identity

$$E'^mK^{-t} = q^{2mt}K^{-t}E'^m, \qquad m,\ t \in \mathbb{N}_0. \qquad \blacksquare$$

3.7. Proof of Theorem 3.5: Since q is not a root of unity, it follows from Lemma 3.3 that there is a coalgebra embedding $\theta : U_q(sl_2) \longrightarrow k\mathcal{D}^c$. Put $E' := K^{-1}E$. Then $\{\ K^lE'^iF^j \mid i,j \in \mathbb{N}_0,\ l \in \mathbb{Z}\ \}$ is a basis of $U_q(sl_2)$. Note that

$$\theta(K^l) = e_l,\ \theta(K^lE') = P_l^{(1)},\ \theta(K^lF) = P_l^{(-1)}.$$

Denote by π the projection $U_q(sl_2) \longrightarrow C(1) \simeq k\mathcal{D}_1$. Then

$$\pi(K^{l-1}E) = P_l^{(1)},\ \pi(K^lF) = P_l^{(-1)},\ \pi(K^lE'^iF^j) = 0 \quad \text{for } i+j \ge 2.$$

By Proposition 2.7(ii) we have

$$\theta(K^lE'^iF^j) = \pi^{\otimes n} \circ \Delta^{n-1}(K^lE'^iF^j)$$

where $n = i+j$, and both i and j are positive integers. By Lemma 3.6 and the definition of π we have

$$\theta(K^lE'^iF^j) = \sum_{s,r} c(s,r)\pi(K^{l-s_1-r_1}E'^{s_0-s_1}F^{r_0-r_1}) \cdot \cdots$$
$$\cdot\, \pi(K^{l-s_{n-1}-r_{n-1}}E'^{s_{n-2}-s_{n-1}}F^{r_{n-2}-r_{n-1}}) \cdot \pi(K^lE'^{s_{n-1}}F^{r_{n-1}})$$

where the dot means the concatenation of paths, and the sum runs over all the vectors $s = (s_0, s_1, \cdots, s_{n-1}),\ r = (r_0, r_1, \cdots, r_{n-1}) \in \mathbb{N}_0^n$, with

$$i = s_0 \ge s_1 \ge \cdots \ge s_{n-1}, \quad j = r_0 \ge r_1 \ge \cdots \ge r_{n-1},$$

such that for each $t,\ 1 \le t \le n$, either

$$s_{t-1} - s_t = 1,\ r_{t-1} - r_t = 0,$$

or

$$s_{t-1} - s_t = 0,\ r_{t-1} - r_t = 1,$$

where s_n and r_n are understood to be zero.

Now, for such a pair (s,r), define $v = (v_1, \cdots, v_n) \in I^n$ as follows:

$$v_{n-t+1} = 1, \quad \text{if } \ s_{t-1} - s_t = 1,\ r_{t-1} - r_t = 0;$$

and

$$v_{n-t+1} = -1, \quad \text{if } \ s_{t-1} - s_t = 0,\ r_{t-1} - r_t = 1,$$

for $1 \le t \le n$. Write (s,r) as $(s,r) = (s_v, r_v)$.

Since $(s_{t-1}+r_{t-1})-(s_t+r_t)=1$ and $s_n+r_n=0$, it follows that $s_t+r_t=n-t$ for $1\leq t\leq n-1$. Therefore, we have

$$\begin{aligned}\theta(K^lE'^iF^j)&=\sum_{s_v,r_v}c(s_v,r_v)P_l^{(v)}\\&=\sum_{v\in I^n,\ |T_v|=i}c(s_v,r_v)P_l^{(v)}.\end{aligned}$$

Note that for $s_v=(i=s_0,s_1,\cdots,s_{n-1})$, any number in the sequence $i-s_1,\ \cdots,\ s_{n-2}-s_{n-1},\ s_{n-1}$ is either 1 or 0, and that the number of 1 in the sequence is exactly i. This implies

$$\binom{i}{s_1}_{q^2}\cdots\binom{s_{n-2}}{s_{n-1}}_{q^2}=i!_{q^2}.$$

In order to compute $c(s_v,r_v)$, let $T_v=\{t_1,\cdots,t_i\}$, with $1\leq t_1<\cdots<t_i\leq n$. By an analysis on the components of

$$r_v=(j=r_0,\cdots,\ r_{n-t_i},\ r_{n-t_i+1},\cdots,\ r_{n-t_{(i-1)}},\cdots,\ r_{t_1},\cdots,\ r_{n-1}),$$

we observe that $r_{n-t_i}=r_{n-t_i+1}$ since $v_{t_i}=1$, and $j=r_0,\cdots,r_{n-t_i}$ are pairwise different. It follows that

$$r_{n-t_i}=j-n+t_i.$$

A similar analysis shows that

$$r_{n-t_x}=j-n+t_x+(i-x),\quad x=1,\cdots,i.$$

It follows that

$$\begin{aligned}\sum_{t=1}^{n-1}r_t(s_{t-1}-s_t)&=\sum_{1\leq t\leq n-1,\ v_{n-t+1}=1}r_t\\&=r_{n-t_1}+\cdots+r_{n-t_i}\\&=(t_1+\cdots+t_i)-\frac{i(i+1)}{2}.\end{aligned}$$

This shows

$$c(s_v,r_v)=i!_{q^2}j!_{q^{-2}}\ q^{-i(i+1)}\chi(v),$$

and hence

$$\begin{aligned}\theta(K^lE'^iF^j)&=i!_{q^2}j!_{q^{-2}}q^{-i(i+1)}\sum_{v\in I^n,\ |T_v|=i}\chi(v)P_l^{(v)}\\&=i!_{q^2}j!_{q^{-2}}q^{-i(i+1)}b(l,n,i)\end{aligned}\tag{3.1}$$

for $n=i+j\geq 2$ and any integer l. Thus $U_q(sl_2)\simeq\theta(U_q(sl_2))$ is spanned by

$$\{b(l,n,i)\mid 0\leq i\leq n,\ n\in\mathbb{N}_0,\quad l\in\mathbb{Z}\},$$

while this set is obviously k-linearly independent. This completes the proof. ■

4. **Comodules of** $U_q(sl_2)$

In this section, by applying Theorem 3.5 we will characterize the category of the $U_q(sl_2)$-comodules in terms of the representations of the quiver $\mathcal{D}$ (see Theorem 4.3), and then list all the indecomposable Schurian $U_q(sl_2)$-comodules (see Theorem 4.7), where q is not a root of unity.

4.1. Let Q be a quiver (not necessarily finite). By definition a k-representation of Q is a datum $V = (V_e, f_a;\ e \in Q_0, a \in Q_1)$, where V_e is a k-space for each $e \in Q_0$, and $f_a : V_{s(a)} \longrightarrow V_{t(a)}$ is a k-linear map for each $a \in Q_1$. Set $f_p := f_{a_l} \circ \cdots \circ f_{a_1}$ for each path $p = a_l \cdots a_1$, where each a_i is an arrow, $1 \leq i \leq l$. Set $f_e := Id$ for $e \in Q_0$. Then f_p is a k-linear map from $V_{s(p)}$ to $V_{t(p)}$. A morphism $\phi : (V_e, f_a;\ e \in Q_0, a \in Q_1) \longrightarrow (W_e, g_a;\ e \in Q_0, a \in Q_1)$ is a datum $\phi = (\phi_e;\ e \in Q_0)$ such that

$$\phi_{t(a)} f_a = g_a \phi_{s(a)}$$

for each $a \in Q_1$. Denote by $\mathrm{Rep}(k, Q)$ the category of the k-representations of Q. We refer the representations of quivers to [ARS] and [Rin].

A representation $V = (V_e, f_a;\ e \in Q_0, a \in Q_1)$ is said to be locally-nilpotent, provided that for each $e \in Q_0$ and each $m \in V_e$, there are only finitely many paths p starting at e such that $f_p(m) \neq 0$.

It was observed by Chin and Quinn that there is an equivalence between the category of the right kQ^c-modules and the category of the locally-nilpotent representations of Q (see [C]). The functors can be seen from the following.

For a right kQ^c-comodule (M, ρ), define for each $e \in Q_0$

$$M_e := \{m \in M \mid \rho_0(m) = m \otimes e\}$$

where $\rho_0 = (Id \otimes \pi_0)\rho$, and $\pi_0 : kQ^c \longrightarrow kQ_0$ is the projection. For every path p there is a unique k-linear map $f_p : M_{s(p)} \longrightarrow M_{t(p)}$, such that for each $m \in M_{s(p)}$ there holds

$$\rho(m) = \sum_{s(p')=s(p)} f_{p'}(m) \otimes p'$$

where p' runs over all the paths with $s(p') = s(p)$. In this way we obtain a k-representation $(M_e, f_a;\ e \in Q_0, a \in Q_1)$ of Q satisfying $f_p = f_\beta f_\alpha$ for any path $p = \beta\alpha$. By construction it is clearly a locally-nilpotent representation. Note that M is a kQ_0-comodule with ρ_0. Since kQ_0 is group-like, it follows that we have a kQ_0-comodule decomposition

$$M = \bigoplus_{e \in Q_0} M_e. \tag{4.1}$$

Conversely, given a locally-nilpotent representation $V = (V_e, f_a;\ e \in Q_0, \alpha \in Q_1)$ of Q, define

$$M := \bigoplus_{e\in Q_0} V_e$$

and $\rho : M \longrightarrow M \otimes kQ^c$ by

$$\rho(m) := \sum_{s(p)=e} f_p(m) \otimes p$$

for each $m \in V_e$ (where f_e is understood to be Id for $e \in Q_0$). Then ρ is well-defined since V is locally-nilpotent and (M, ρ) is a right kQ^c-comodule.

4.2. Keep the notations in 3.2. Given a representation $V = (V_l, V_a;\ e_l \in \mathcal{D}_0, a \in \mathcal{D}_1)$ of the quiver $\mathcal{D}$, define $f_l^{(v)} := f_{P_l^{(v)}}$, for each integer l and $v \in \mathcal{I}$. In particular, $f_l^{(0)} = Id$.

With the help of the representations of a quiver and Theorem 3.5, we can describe the category of the comodules of $U_q(sl_2)$.

Theorem 4.3. *Assume that q is not a root of unity. Then there is an equivalence between the category of the right $U_q(sl_2)$-comodules and the full subcategory of* $\mathrm{Rep}(k, \mathcal{D})$ *whose objects $V = (V_l,\ f_a :\ e_l \in \mathcal{D}_0,\ a \in \mathcal{D}_1)$ satisfies the following conditions:*

(i) $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$ *for all* $l \in \mathbb{Z}$.

(ii) *For any $m \in V_l$, $f_l^{(v)}(m) = 0$ for all but finitely many $v \in \mathcal{I}$.*

Proof By Theorem 3.5, as a coalgebra $U_q(sl_2)$ is isomorphic to the subcoalgebra $\mathcal{C}$ of path coalgebra $k\mathcal{D}^c$ with the set of basis

$$\{b(l, n, i) := \sum_{v\in I^n,\ |T_v|=i} \chi(v) P_l^{(v)} \mid 0 \le i \le n,\ n \in \mathbb{N}_0,\ l \in \mathbb{Z}\}.$$

For a coalgebra C, let $\mathcal{M}^C$ denote the category of the right C-comodules. So we have the following embedding of categories

$$\mathcal{M}^{U_q(sl_2)} \simeq \mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \mathrm{Rep}(k, \mathcal{D}),$$

where $\mathcal{M}^{\mathcal{C}} \hookrightarrow \mathcal{M}^{k\mathcal{D}^c}$ since $\mathcal{C}$ is a subcoalgebra of $k\mathcal{D}^c$, and $\mathcal{M}^{k\mathcal{D}^c} \hookrightarrow \mathrm{Rep}(k, \mathcal{D})$ is the embedding described in 4.1.

Now, the question is reduced to determine all locally-nilpotent k-representations of quiver $\mathcal{D}$ which are right $\mathcal{C}$-comodules, via the equivalence described in 4.1.

It follows from the definition that a representation $V = (V_l, f_a : e_l \in \mathcal{D}_0,\ a \in \mathcal{D}_1)$ of quiver $\mathcal{D}$ is locally-nilpotent if and only if the condition (ii) is satisfied. Assume that such a V is locally-nilpotent, then $M = \bigoplus_{l\in\mathbb{Z}} V_l$ becomes a right $k\mathcal{D}^c$-comodule via

$$\rho(m) = \sum_{v\in\mathcal{I}} f_l^{(v)}(m) \otimes P_l^{(v)} \in M \otimes k\mathcal{D}^c$$

for all $m \in V_l,\ l \in \mathbb{Z}$.

If for an arbitrary fixed $m \in V_l,\ l \in \mathbb{Z}$, the element $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$, then we can write

$$\begin{aligned}\rho(m) &= \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n}\sum_{v\in I^n,\ |T_v|=i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v)P_l^{(v)} \\ &= \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes (\sum_{v\in I^n,\ |T_v|=i} \chi(v)P_l^{(v)}) \\ &\in M\otimes \mathcal{C},\end{aligned}$$

and hence M becomes a right $\mathcal{C}$-comodule. Conversely, if M becomes a right $\mathcal{C}$-comodule, then we have

$$\begin{aligned}\rho(m) &= \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n}\sum_{v\in I^n,\ |T_v|=i} \frac{f_l^{(v)}(m)}{\chi(v)} \otimes \chi(v)P_l^{(v)} \\ &= \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n} m(n,i) \otimes (\sum_{v\in I^n,\ |T_v|=i} \chi(v)P_l^{(v)})\end{aligned}$$

for some $m(n,i)\in M$. Since

$$\{\chi(v)P_l^{(v)} \mid v\in I^n,\ |T_v|=i,\ 0\le i\le n,\ n\in\mathbb{N}_0,\ l\in\mathbb{Z}\}$$

is a set of basis of $k\mathcal{D}^c$, it follows that

$$m(n,i) = \frac{f_l^{(v)}(m)}{\chi(v)},$$

which implies that $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$ for an arbitrary fixed $m \in V_l,\ l\in\mathbb{Z}$.

Now, the condition (i) implies that for an arbitrary fixed $m\in V_l,\ l\in\mathbb{Z}$, the element $\frac{f_l^{(v)}(m)}{\chi(v)}$ only depends on $|v|$ and $|T_v|$. Conversely, by taking $v=(-1,1)$ and $v'=(1,-1)$ in $\mathcal{I}$ we obtain

$$\frac{f_l^{(-1,1)}(m)}{\chi((-1,1))} = \frac{f_l^{(1,-1)}(m)}{\chi((1,-1))},$$

which is exactly the condition (i). This completes the proof. ■

Theorem 4.3 permits us to explicitly construct some $U_q(sl_2)$-comodules. In the following q is not a root of unity.

Example 4.4. *Let A be the quantum plane generated by X and Y subject to the relation $XY = q^2YX$. Let l be an integer and n a non-negative integer. Then for any A-module U one can define a representation $V = V_{(l,n,U)}$ of quiver $\mathcal{D}$ as follows:*

$$\begin{aligned}
&V_j := U, && \textit{if } \ l \le j \le l+n,\\
&V_j := 0, && \textit{otherise};\\
&f_j^{(1)} := X, && \textit{if } \ l+1 \le j \le l+n,\\
&f_j^{(1)} := 0, && \textit{otherwise};\\
&f_j^{(-1)} := Y, && \textit{if } \ l+1 \le j \le l+n,\\
&f_j^{(-1)} := 0 && \textit{otherwise}.
\end{aligned}$$

where l is any integer and $n \geq 0$. Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule.

Example 4.5. *Let l be an integer and n a non-negative integer.*

(i) For each $\lambda \in k$, one can define a representation V of quiver $\mathcal{D}$ as follows:

$$\begin{aligned}
&V_j := k, && \textit{if } \ l \le j \le l+n,\\
&V_j := 0, && \textit{otherwise};\\
&f_j^{(1)} := 1, && \textit{if } \ l+1 \le j \le l+n,\\
&f_j^{(1)} := 0, && \textit{otherwise};\\
&f_j^{(-1)} := \lambda q^{-2(l+n-j)}, && \textit{if } \ l+1 \le j \le l+n,\\
&f_j^{(-1)} := 0, && \textit{otherwise}.
\end{aligned}$$

Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule, which is denoted by $M_{(l,n,\lambda)}$.

(ii) Consider the representation V of quiver $\mathcal{D}$ defined by:

$$\begin{aligned}
&V_j := k, && \textit{if } \ l \le j \le l+n,\\
&V_j := 0, && \textit{otherwise};\\
&f_j^{(1)} := 0, && \forall \ j \in \mathbb{Z};\\
&f_j^{(-1)} := 1, && \forall j \in \mathbb{Z}.
\end{aligned}$$

Then by Theorem 4.3, V induces a right $U_q(sl_2)$-comodule, which is denoted by $M_{(l,n,\infty)}$.

4.6. A finite-dimensional right $U_q(sl_2)$-comodule (M, ρ) is said to be Schurian, if $\dim_k M_j = 1$ or 0 for each integer j, where $M_j := \{m \in M \mid (Id \otimes \pi_0)\rho(m) = m \otimes e_j\}$ and π_0 is the projection from $k\mathcal{D}^c$ to $k\mathcal{D}_0$.

Theorem 4.7. *When the triple (l, n, λ) runs over $\mathbb{Z} \times \mathbb{N}_0 \times (k \cup \{\infty\})$, $M_{(l,n,\lambda)}$ gives a complete list of all pairwise non-isomorphic, indecomposable Schurian right $U_q(sl_2)$-comodules, where q is not a root of unity.*

Proof Assume that M is an indecomposable Schurian right $U_q(sl_2)$-comodule. Set $\mathrm{Supp}(M) := \{j \in \mathbb{Z} \mid M_j \neq 0\}$. Let l and $l+n$ be the minimal and the maximal elements in $\mathrm{Supp}(M)$. Then $\mathrm{Supp}(M) \subseteq \{l, l+1, \cdots, l+n\}$. We claim that $\mathrm{Supp}(M) = \{l, l+1, \cdots, l+n\}$.

Otherwise, there exist a j_0 such that $l < j_0 < l+n$ and $j_0 \notin \mathrm{Supp}(M)$. Then by (4.1) we have a $k\mathcal{D}_0$-comodule decomposition

$$M = (\bigoplus_{j<j_0} M_j) \bigoplus (\bigoplus_{j>j_0} M_j).$$

Since $M_{j_0} = 0$, it follows that this is a $k\mathcal{D}^c$-comodule decomposition, and hence it is also a $U_q(sl_2)$-comodule decomposition, which contradicts to the assumption.

Note that each M_j is one-dimensional for $l \le j \le l+n$. Set

$$a_j := f_j^{(1)} \quad \text{and} \quad b_j := f_j^{(-1)}, \quad l+1 \le j \le l+n.$$

Note that for each j, we have $a_j \neq 0$ or $b_j \neq 0$ (otherwise, say $a_{j_0} = b_{j_0} = 0$, then we again have a $U_q(sl_2)$-comodule decomposition $M = (\oplus_{j<j_0} M_j) \oplus (\oplus_{j \ge j_0} M_j)$).

By Theorem 4.3 we have $a_j b_{j+1} = q^2 b_j a_{j+1}$ for all j with $l+1 \le j \le l+n-1$. Now, if some $b_{j_0} = 0$, then all $b_j = 0$ and all $a_j \neq 0$, and hence M is isomorphic to $M_{(l,n,0)}$. If some $a_{j_0} = 0$, then all $a_j = 0$ and all $b_j \neq 0$, and hence M is isomorphic to $M_{(l,n,\infty)}$. If $a_j \neq 0 \neq b_j$ for all j, then M is isomorphic to $M_{(l,n,\lambda)}$ with $\lambda = \frac{b_{l+1}}{a_{l+1}} q^{2(n-1)}$.

On the other hand, each $M_{(l,n,\lambda)}$ is indecomposable since its socle is of one dimension, and they are clearly pairwise non-isomorphic. ∎

5. **A class of $SL_q(2)$-modules**

Theorem 4.3 characterizes the category of the right $U_q(sl_2)$-comodules by a full subcategory of the category of the k-representations of $\mathcal{D}$, where q is not a root of unity, and $\mathcal{D}$ is the Gabriel quiver of $U_q(sl_2)$ as a coalgebra. This permits us to construct some left $SL_q(2)$-modules from some representations of quiver $\mathcal{D}$, via the duality between $U_q(sl_2)$ and $SL_q(2)$.

5.1. Recall that the algebra homomorphism $\psi : SL_q(2) \longrightarrow U_q(sl_2)^*$ in Lemma 2.3 is given by

$$\psi(a)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,0} q^l + \delta_{i,1}\delta_{j,1} q^{l-1}, \qquad \psi(b)(K^l E'^i F^j) = \delta_{i,1}\delta_{j,0} q^{l-1},$$
$$\psi(c)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,1} q^{-l}, \qquad \psi(d)(K^l E'^i F^j) = \delta_{i,0}\delta_{j,0} q^{-l},$$

where $E' = K^{-1}E$.

Let (M, ρ) be a right $U_q(sl_2)$-comodule. Then M becomes a left $SL_q(2)$-module via

$$x.m := \sum \psi(x)(m_1) m_0, \tag{5.1}$$

for $x \in SL_q(2),\ m \in M$, where $\rho(m) = \sum m_0 \otimes m_1 \in M \otimes U_q(sl_2)$.

Let $\mathcal{C}$ be the subcoalgebra of $k\mathcal{D}^c$ with the set of basis

$$\{b(l,n,i) = \sum_{v\in I^n,\ |T_v|=i} \chi(v)P_l^{(v)} \mid 0\le i\le n,\quad n\in\mathbb{N}_0,\quad l\in\mathbb{Z}\}.$$

Identifying $U_q(sl_2)$ with $\mathcal{C}$ via (3.1), we can evaluate $\psi(a)$, $\psi(b)$, $\psi(c)$ and $\psi(d)$ on this set of basis of $\mathcal{C}$ via Lemma 2.3. Since

$$b(l,n,i) = \frac{q^{i(i+1)}}{i!_{q^2}j!_{q^{-2}}}\theta(K^lE'^iF^j) \ \text{ with } \ j=n-i,$$

it follows that the list of the non-zero values is as follows:

$$\begin{aligned}
\psi(a)(b(l,0,0)) &= \psi(a)(K^l) = q^l,\\
\psi(a)(b(l,2,1)) &= \psi(a)(q^2K^lE'F) = q^{l+1},\\
\psi(b)(b(l,1,1)) &= \psi(b)(q^2K^lE') = q^{l+1},\\
\psi(c)(b(l,1,0)) &= \psi(c)(K^lF) = q^{-l},\\
\psi(d)(b(l,0,0)) &= \psi(d)(K^l) = q^{-l}.
\end{aligned}$$

Theorem 5.2. *Let $V=(V_l,\ f_a:\ l\in\mathcal{D}_0,\ a\in\mathcal{D}_1)$ be a k-representation of the quiver $\mathcal{D}$ satisfying the following conditions:*

(i) $f_{l-1}^{(1)}\circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)}\circ f_l^{(1)}$ *for all* $l\in\mathbb{Z}$.

(ii) For any $m\in V_l$, $f_l^{(v)}(m)=0$ for all but finitely many $v\in\mathcal{I}$, where $f_l^{(v)} = f_{P_l^{(v)}}$, $f_l^{(0)} = Id|_{V_l}$.

Then $M=\bigoplus_{l\in\mathbb{Z}} V_l$ is a left $SL_q(2)$-module via

$$\begin{aligned}
a.m &= q^lm + q^{l-1}f_l^{(1,-1)}(m),\\
b.m &= q^{l-1}f_l^{(1)}(m),\\
c.m &= q^{-l}f_l^{(-1)}(m),\\
d.m &:= q^{-l}m,
\end{aligned}$$

for each $m\in V_l,\ l\in\mathbb{Z}$, where q is not a root of unity.

Proof By Theorem 4.3 $M=\bigoplus_{l\in\mathbb{Z}} V_l$ is a right $U_q(sl_2)$-comodule via

$$\rho(m) = \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n}\frac{f_l^{(v)}(m)}{\chi(v)}\otimes b(l,n,i)$$

where v is a fixed element in I^n with $|T_v|=i$, and $m\in V_l,\ l\in\mathbb{Z}$. By (5.1), M becomes a left $SL_q(2)$-module via

$$x.m := \sum_{n\in\mathbb{N}_0}\sum_{0\le i\le n}\psi(x)(b(l,n,i))\frac{f_l^{(v)}(m)}{\chi(v)}.$$

It follows that for each $m \in V_l,\ l \in \mathbb{Z}$, we have

$$\begin{aligned}
a.m &= \psi(a)(b(l,0,0))m + \psi(a)(b(l,2,1))\frac{f_l^{(1,-1)}(m)}{\chi(1,-1)} \\
&= q^l m + q^{l-1} f_l^{(1,-1)}(m), \\
b.m &= \psi(b)(b(l,1,1))\frac{f_l^{(1)}(m)}{\chi(1)} = q^{l-1} f_l^{(1)}(m), \\
c.m &= \psi(c)(b(l,1,0))\frac{f_l^{(-1)}(m)}{\chi(0)} = q^{-l} f_l^{(-1)}(m), \\
d.m &:= \psi(d)(b(l,0,0))m = q^{-l} m.
\end{aligned}$$

■

Theorem 5.2 permits us to write out explicitly the following examples of $SL_q(2)$-modules.

Example 5.3. *Let A be the quantum plane generated by X and Y subject to the relation $XY = q^2 YX$, and U be a left A-module, where q is not a root of unity. Let l be an integer and n a non-negative integer. For any element $u \in U$ and $1 \le i \le n+1$, let U^{n+1} denote the direct sum of the copies of U, and u_i denote the element in U^{n+1} with the i-th component being u and other components being 0. Then by Theorem 5.2 and Example 4.4, the copy U^{n+1} becomes a left $SL_q(2)$-module with the following actions :*

$$\begin{array}{llll}
au_i = q^{i+l-1}u_i + q^{i+l-2}YXu_{i-2}, & 3 \le i \le n+1, & au_i = 0, & \textit{otherise}, \\
bu = q^{i+l-2}Xu_{i-1}, & 2 \le i \le n+1, & bu_i = 0, & \textit{otherise}, \\
cu_i = q^{-(i+l-1)}Yu_{i-1}, & 2 \le i \le n+1, & cu_i = 0, & \textit{otherise}, \\
du_i = q^{-(i+l-1)}u_i, & \forall i \quad . & &
\end{array}$$

Example 5.4. *Let V be a k-space of dimension $n+1$, $n \in \mathbb{N}_0$, with basis $v_0, v_1, \cdots, v_n$. Let l be an integer, and $q \in k$ be not a root of unity.*

(i) Let $\lambda \in k$. Then by Theorem 5.2 and Example 4.5(i), V becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\lambda)}$

$$\begin{array}{lr}
a.v_i = q^{l+i}v_i + \lambda q^{-2n+l+3i-3}v_{i-2}, & 2 \le i \le n, \\
a.v_i = q^{l+i}v_i, & i = 0,1, \\
b.v_i = q^{l+i-1}v_{i-1}, & 1 \le i \le n, \\
b.v_0 = 0, & \\
c.v_i = \lambda q^{-2n+i-l}v_{i-1}, & 1 \le i \le n, \\
c.v_0 = 0, & \\
d.v_i = q^{-(l+i)}v_i, & \forall\ i.
\end{array}$$

(ii) By Theorem 5.2 and Example 4.5(ii), V also becomes a left $SL_q(2)$-module via the following actions, which is denoted again by $M_{(l,n,\infty)}$

$$\begin{aligned}
a.v_i &= q^{l+i}v_i, && \forall\ i,\\
b.v_i &= 0, && \forall\ i,\\
c.v_i &= q^{-(l+i)}v_{i-1}, && 1 \le i \le n,\\
c.v_0 &= 0,\\
d.v_i &= q^{-(l+i)}v_i, && \forall\ i.
\end{aligned}$$

Note that $M_{(l,n,\lambda)}$ with $l \in \mathbb{Z},\ n \in \mathbb{N}_0,\ \lambda \in k \cup \{\infty\}$ are indecomposable, pairwise non-isomorphic $SL_q(2)$-modules.

References

[ARS] M. Auslander, I. Reiten, and S.O. Smalϕ, Representation Theory of Artin Algebras, Cambridge Studies in Adv. Math. vol. 36, Cambridge Univ. Press, 1995.

[B] I. Boca, The coradical filtration of $U_q(sl_2)$ at root of unity, Commun. Algebra 22(14)(1994), 5769-5776.

[C] W. Chin, A brief introduction to coalgebra representation theory, In: Hopf algebras (DePaul Conf. Proc.), Lecture Notes in Pure and Appl. Math., vol. 237, 109-131, Marcel Dekker Inc., New York, 2004.

[CHYZ] X.W. Chen, H.L. Huang, Y. Ye, and P. Zhang, Monomial Hopf algebras, J. Algebra 275(2004), 212-232.

[CHZ] X.W. Chen, H.L. Huang, and P. Zhang, Dual Gabriel theorem with applications, preprint, 2004.

[CK] W. Chin and L. Krop, Injective comodules for 2×2 quantum matrices, Commun. Algebra 28(4)(2000), 2043-2057.

[CMon] W. Chin and S. Montgomery, Basic coalgebras, In: Modular interfaces (Reverside, CA, 1995), 41-47, AMS/IP Stud. Adv. Math. 4, Amer. Math. Soc., Providence, RI, 1997.

[CMus] W. Chin and I. Musson, The coradical filtration of quantum enveloping algebras, J. London Math. Soc. 53(2)(1996), 50-62.

[CR] C. Cibils and M. Rosso, Hopf quivers, J. Algebra 254(2002), 241-251.

[D] Y. Doi, Homological coalgebras, J. Math. Soc. Japan 33(1)(1981), 31-50.

[Dr] V.G. Drinfeld, Quantum group, In: Proc. ICM Berkeley, 798-820, Amer. Math. Soc., Providence, RI, 1986.

[G] J. A. Green, Locally finite representations, J. Algebra 41(1976), 137-171.

[J] J. C. Jantzen, Lectures on Quantum Group, Graduate Studies in Math. 6, Amer. Math. Soc., Providence, RI, 1996.

[K] C. Kassel, Quantum Group, Graduate Texts in Math. 155, Springer-Verlag, Berlin, Heidelberg, New York, 1995.

[L] G. Lusztig, Introduction to quantum groups, Progress in Math. vol. 110, Birkhäuser, Boston, 1993.

[M1] S. Montgomery, Some remarks on filtrations of Hopf algebras, Commun. Algebra 21(1993), 999-1007.

[M2] S. Montgomery, Hopf Algebras and Their Actions on Rings, CBMS Regional Conf. Series in Math. 82, Amer. Math. Soc., Providence, RI, 1993.

[M3] S. Montgomery, Indecomposable coalgebras, simple comodules and pointed Hopf algebras, Proc. Amer. Math. Soc. 123(1995), 2343-2351.

[Mü] E. Müller, The coradical filtration of $U_q(g)$ at root of unity, Commun. Algebra 28(2)(2000), 1029-1044.

[OZ] F. van Oystaeyen, and P.Zhang, Quiver Hopf algebras, J. Algebra 280(2004), 577-589.

[PW] B. Parshall and J.P. Wang, Quantum Linear Groups, Mem. Amer. Math. Soc. vol. 439, Amer. Math. Soc., Providence, RI, 1991.

[Rad] D. E. Radford, On the structure of pointed coalgebras, J. Algebra 77(1)(1982), 1-14.
[Rin] C. M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math. 1099, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.
[Ro] M. Rosso, Finite-dimensional representations of the quantum analog of the enveloping algebras of a complex Lie algebra, Commun. Math. Phys. 117(1988), 561-593.
[Sim] D. Simson, On coalgebras of tame comodule type, In: Proc. the 9-th ICRA, vol.2, 450-486, Beijing Normal Univ. Press, 2000.
[Sw] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
[T] M. Takeuchi, Some topics on $GL_q(n)$, J. Algebra 147(1992), 379-410.